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Magnetic Monopoles and the Orientation Entanglement Relation

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We show that Dirac magnetic monopoles do not satisfy the orientation entanglement (OE) relation and do not return to their initial state after a 4π rotation. This is done in both the formalism using Dirac strings of singularities and in the fiber bundle formalism. In the latter we connect the OE relation to the first homotopy group of the gauge group. We hypothesize that failure to satisfy the OE relation is the reason Dirac magnetic monopoles have never been seen.

1. INTRODUCTION

The orientation entanglement (OE) relation has been described by McDonald [1]. A body is connected to the corners of a room using elastic cords. The body is rotated through 4π and then held fixed. McDonald has then shown that by looping the elastic cords around the body in a sequence of manipulations show in detail in Misner *et al.* [2] the twists in the elastic cords can be completely undone. The body and the elastic cords are returned to their initial configuration before the 4π rotation. It is well known that a spinor returns to its initial configuration after a 4π rotation, but not after a 2π rotation, where it picks up a minus sign. The OE relation represents a deeper connection to our environment than the usual geometry would suggest. Spinors sense this deeper connection in some way that vectors do not (we look at this in much more detail below when discussing SO_3).

In this paper we hypothesize that the OE relations are important to physics. They represent the deep relationship between any particle or material body and its environment. We thus hypothesize that any free particle must return to its initial configuration after a 4π rotation if it is to exist and that

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any particle which does not satisfy the OE relation cannot exist. So far all particles which have been observed in physics satisfy the OE relation. We will show below, using two different methods, that Dirac [3] magnetic monopoles do not satisfy the OE relation, and we hypothesize that this is the reason they have never been seen despite extensive searches [4–11] and despite having a natural and elegant theory underlying them [12–20], going back to the more natural symmetry of Maxwell's equations with magnetic monopole sources present. Since all known particles satisfy the OE relations and we show that Dirac magnetic monopoles which have not been seen do not satisfy these relations, it is hoped that this paper will stimulate further work on the OE relations themselves and on their topological role in physics. This paper does not prove that Dirac magnetic monopoles cannot exist, but does point out a suggestive relationship between Dirac magnetic monopoles and the OE relations. We will be primarily interested in the magnetic monopoles first described in the classic papers of Dirac [3] and more recently in the work of Wu and Yang [21]. 't Hooft [22]–Polyakov [23] monopoles are different and will be discussed more in Section 3.

2. DIRAC MAGNTIC MONOPOLES DO NOT SATISFY THE OE RELATION

Let us first look at the description of magnetic monopoles proposed by Dirac [3]. To write down the wave equation of an electron in the presence of a magnetic monopole, one needs the vector potential associated with the pole. Dirac constructs the vector potential by considering the pole as the endpoint of a string of magnetic dipoles whose other end is at infinity. If the pole is at rest at the origin, we can take the string along the negative z axis and write

$$A_x = \frac{-gy}{r(r+z)}, \qquad A_y = \frac{gx}{r(r+z)}, \qquad A_z = 0 \tag{1}$$

with $r = (x^2 + y^2 + z^2)^{1/2} > 0$. This vector potential is singular along the string of singularities ($z \to -r$) and has a curl which gives $\vec{H} = g\vec{r}/r^3$, the field of our monopole. A vector potential which is nonsingular everywhere and is defined on a single coordinate patch does not exist, as we will see in the better treatment using fiber bundles below. Dirac showed in his paper how to handle the string of singularites attached to the monopole, and a self-consistent theory of electrons interacting with magnetic monopoles resulted so long as the Dirac quantization condition

$$\frac{eg}{\hbar c} = \frac{n}{2} \tag{2}$$

with n an integer, was satisfied.

Now let us look at the OE relation. In this Dirac string picture a monopole always comes with a semiinfinite string of singularities in A_μ attached. If we attach elastic cords to the monopole and rotate the monopole by 4π, the string of singularities does not wrap up, but ends on the monopole exactly as before. Now we can hold the monopole fixed and try to loop the elastic cords around the monopole as in the McDonald manipulation above. We cannot do this without cutting the string of singularities with the cord. But this is forbidden topologically because the string of singularities is really a part of the magnetic monopole itself in this description. Thus the magnetic monopole with string of singularities attached as in Dirac's theory does not satisfy the OE relation.

This argument is reasonable, but not completely rigorous because of the prohibition that the elastic cords not cut the string of singularities. By going to a description of the magnetic monopole in terms of a nontrivial fiber bundle, we can make the argument more rigorous and arrive at the relationship between satisfying the OE relation and the homotopic classification of the bundle. Wu and Yang [21] first described magnetic monopoles in terms of a principal fiber bundle with a nontrivial connection A_μ (the vector potential of gauge theory). In general a principal fiber bundle locally is a product of a base space (often space-time) and a structure group (the gauge group of gauge theory). Globally, however, there can be nontrivial twists in the topology of the bundle. The reader is referred to the literature for a rigorous mathematical exposition of fiber bundles [24–30]. Following Wu and Yang [21], we can consider a static magnetic monopole and take the base space to be $R_3 - (0)$, which retracts to S_2. The gauge group is the U_1 gauge group for electromagnetism, isomorphic to S_1. Since we have a nontrivial bundle when magnetic monopoles are present, A_μ must be defined separately on each of two open sets covering the base space S_2. Wu and Yang [21] give

$$
\begin{aligned}
A_{\phi_a} &= \frac{g}{r \sin\theta}(1 - \cos\theta) \\
A_{\phi_b} &= \frac{-g}{r \sin\theta}(1 + \cos\theta)
\end{aligned}
\tag{3}
$$

where set a is $0 \le \theta < (\pi/2) + \delta$, $0 \le \phi < 2\pi$, and set b is $(\pi/2) - \delta < \theta \le \pi$, $0 \le \phi < 2\pi$. Now, A_{ϕ_a} and A_{ϕ_b} must be related by a gauge transformation,

$$
A_{\mu_b} = A_{\mu_a} + \frac{\hbar c}{e}\frac{\partial\alpha}{\partial x^\mu}
\tag{4}
$$

in the overlap region $(\pi/2) - \delta < \theta < (\pi/2) + \delta$, $0 \le \phi < 2\pi$. This overlap region is a band around the equator of S_2. Under this gauge transformation an electron wave function will change according to

$$\Psi_b = e^{i\alpha}\,\Psi_a \tag{5}$$

From (3) and (4) we have that the function α is

$$\alpha = \frac{-2eg\phi}{\hbar c} \tag{6}$$

Requiring the electron wave function to be single-valued in the overlap region then gives the Dirac [3] quantization condition (2). If we look at an electric charge rather than a magnetic monopole, then we have a trivial bundle. In this case, A_μ can be defined by the same expression everywhere and two different open sets are not required.

Steenrod [24] has shown that $\Pi_1[G]$ classifies a fiber bundle with structure group G and base space S_2, where Π_1 is the first homotopy group. For electromagnetism this is $\Pi_1[U_1] = ZZ$, where ZZ is the additive group of integers. The integer appearing in this classifacation is the same as the integer appearing in the quantization condition (2). This nontrivial homotopy is really associated with maps from the overlap region (which retracts to S_1) to the gauge group (U_1) and shows that the bundles associated with magnetic monopoles really are nontrivial. The transition function is the $e^{i\alpha}$ factor in (5), which becomes $e^{-in\phi}$ using (2) and (6), with n an integer. As ϕ goes from 0 to 2π in the overlap region, this transition function clearly wraps around U_1 a total of n times.

Now how does this fiber bundle description of magnetic monopoles relate to the OE relation? Imagine the two hemispheres (the sets a and b above) used to describe A_μ in (3) attached to the corners of a room with eight elastic cords. Four cords A, B, C, D attach the upper hemispheres to the upper four corners of the room and four cords E, F, G, H attach the lower hemisphere to the lower four corners of the room. Now the crucial points are that two different coordinate patches are *required* to define a singularity free A_μ for a magnetic monopole and that we can rotate the hemispheres a and b independently about the polar axis (which we are free to choose). Since A_μ in (3) depends only on r and θ, this independent rotation of the two hemisphres does not change A_μ. Now imagine rotating the top hemisphere relative to the stationary particle, leaving the bottom hemisphere untwisted. The McDonald manipulation of the elastic cords in the OE relation mentioned above cannot undo this relative twist of the two hemispheres if the body itself is now rotated through 4π or any other angle. Thus a Dirac magnetic monopole described by (3) does not satisfy the OE relation. We then hypothesize as above that Dirac magnetic monopoles cannot exist. Their failure to return to their original configuration after a 4π rotation is the reason they have never been seen.

Note that when we look at the two coordinate patches necessary to define A_μ above, these are two overlapping open sets on the S_2 of the base space. This S_2 is the retract of the space $R_3 - 0$ in which the static monopole lives. We see then the relationship between this S_2 of the fiber bundle description and the "room" in the OE relation to which the elastic cords are attached.

We can generalize our result a bit. $\Pi_1[G]$ in general classifies a bundle with structure group G and base space S_2 (appropriate for static particles) [24]. If (case I) $\Pi_1[G] = 0$, the fiber bundle is trivial and only one coordinate patch is necessary to describe the particle. Our above analysis then says that the OE relations can be satisfied. If (case II) $\Pi_1[G] \neq 0$, we need to look at the bundles on a case by case basis. We saw above that for magnetic monopoles $\Pi_1[U_1] = ZZ$ and the OE relation was not satisfied. The fiber bundle for an ordinary charged particle has U_1 for the structure group, but has $n = 0$ in the Dirac quantization condition (2) and corresponds to trivial homotopic maps and to a trivial fiber bundle. A_μ for an ordinary charged particle requires only one coordinate patch on the S_2 retracted base space for its definition. Thus we need to look at the structure of the bundle describing the particle to see whether the OE relation can be satisfied or not.

As a further simple example, consider the rotation group SO_3 with homotopic classification $\Pi_1[SO_3] = Z_2$ the group of integers modulo 2. Here the Z_2 separates rotations into two homotopy classes. In one class we have vectors and tensors and in the other we have spinors which return to their initial configuration after a 4π rotation, but not after a 2π rotation. A spinor is a double covering of the base space and is described by a nontrivial bundle. Spinors explicitly satisfy the OE relation, and ordinary vectors or tensors certainly return to their initial configuration also after a 4π rotation, so particles with either type spin satisfy the OE relation. Thus even though $\Pi_1[SO_3] \neq 0$, the OE relation is satisfied. If $\Pi_1[G] \neq 0$ (case II), we must look at the bundles on a case by case basis.

3. DISCUSSION

We have been discussing Dirac magnetic monopoles in this paper. 't Hooft [22]–Polyakov [23] monopoles are quite different. These are extended objects with finite and calculable energy arising from spontaneous symmetry breaking. Ezawa and Tze [31] have shown that these are characterized by $\Pi_1[H]$, where H is an isotropy subgroup of the original gauge group G. 't Hooft shows that for the spontaneous symmetry breaking of SO_3 to U_1 we can have a magnetic monopole. The key is that the fiber bundle is still characterized by $\Pi_1[SO_3] = Z_2$ and this bundle can satisfy the OE relation as we saw above in a different context. The spontaneous symmetry breaking gives us the U_1 necessary for an electromagnetic magnetic monopole, but SO_3 is still present in a crucial way. Not surprisingly, 't Hooft showed that no Dirac string of singularities is necessary. Ezawa and Tze [31] explicitly showed that only a single coordinate patch is required for this monopole. Thus both our arguments above would say that this type of magnetic monopole can satisfy the OE relation. Thus we cannot use the OE relation to rule them out. They do rely upon spontaneous symmetry breaking, however, which itself has not been verified experimentally in high-energy physics.

Wu and Yang [21] also discuss Dirac-type SU_2 monopoles. These do not carry a U_1 electromagnetic field and therefore are not really magnetic monopoles. These monopoles are represented by a trivial fiber bundle since $\Pi_1[SU_2] = 0$. Only one coordinate patch is necessary to define the connection. Wu and Yang [21] find only one type of such particle with nothing analogous to the Dirac quantization condition (2). This agrees with our work above. These monopoles are the analogue of charged particles in electromagnetism except they carry SU_2 charge. They are not ruled out by the OE relation.

As a final comment, electromagnetism is invariant under a duality rotation under which the electric and magnetic fields and sources exchange roles [2]. We can go to a dual picture where Dirac magnetic monopoles are described by a trivial connection and ordinary charged particles by a nontrivial one. In this theory magnetic monopoles could exist, but not ordinary electric monopoles. Since we know that electric monopoles exist, we use the version of the theory where magnetic monopoles are described by a nontrivial bundle which does not satisfy the OE relation. We have really shown that *both* electric and magnetic monopoles cannot exist, only one or the other. One can also see this from the Dirac quantization condition (2). To satisfy the OE relation we must have a trivial bundle and $n = 0$. Thus if $e \neq 0$, then $g = 0$. If $g \neq 0$, then $e = 0$.

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